# 1

# NUMERICAL METHODS

## 1.1 Introduction

In this chapter of Numerical methods, we shall see the methods of solving Algebraic and trancendental equations, simultaneous linear equations, ordinary differential equations, numerical integration, also interpolation.

## 1.2 Algebraic and Transcendental equations

For the benefit of the readers we mention below few basic concepts of theory of equations related to algebraic equations.

An expression of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots\dots\dots + a_n ,$$

where $n$ is a positive integer and $a_0, a_1, a_2, \dots\dots a_n$a are real constants, $a_0 \neq 0$, is called a **polynomial in $x$ of degree $n$.**

If $f(x)$ is a polynomial of degree n, then $f(x) = 0$, is called an **algebraic equation of degree $n$.**

The algebraic equations of degree 1, 2, 3 and 4 are respectively called *linear, quadratic, cubic* and *biquadratic* equations.

A real or complex number '$a$' is called a *root* of the equation $f(x) = 0$, if $f(a) = 0$.

Following are few results related to the roots of the equation $f(x) = 0$.

**1.** Every equation $f(x) = 0$ of degree $n > 0$, has atleast a root, real or imaginary. This statement is known as the "**Fundamental theorem of algebra**".

**2.** Every equation $f(x) = 0$ of degree $n > 0$ has atmost $n$ roots.

**3.** If '$a$" is a root of the equation $f(x) = 0$, of degree $n$, then $(x - a)$ is a factor of $f(x)$. That is $f(x)$ can be expressed as $f(x) = (x - a) . g(x)$, where $g(x)$ is a polynomial of degree $(n - 1)$.

If $f(x) = ax^2 + bx + c = 0$, we have a simple formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to obtain the roots.

Similarly, there are few methods of solving cubic equation. However for higher degree polynomial equations, it is difficult and in many cases impossible to get closed form of solutions.

Apart from the algebraic equations, we have another type of equations called **transcendental equation.** These are the equations in which transcendental functions such as $\sin x$, $\cos x$, $\log x$, $e^x$ etc involve.

The following are few examples of algebraic and transcendental equations.

$$x^2 + 3x + 2 = 0, \quad 3x^3 - 4x^2 + 2x + 1 = 0, \quad x^5 + 4x^3 + 2x + 1 = 0$$

are few algebraic equations.

$$3e^x + 1 = 0, \quad \cos x + 2x = 1, \quad \log x - 2x = 12$$

are few transcendental equations.

Due to limitations of analytical methods, formula giving exact roots of the equations exist only for lower degree algebraic equations and in very simple cases of transcendental equations. For higher degree polynomial equations and in the case of transcendental equations, it is difficult and in many cases impossible to get closed form solution. Therefore we have to use "approximate" methods, called *numerical methods,* to solve such algebraic and transcendental equations. In this section we shall discuss few such methods.

The numerical methods usually consists in repeated application of the same process, where at each step the result obtained in the previous step is used. Such process is called "*iteration*" methods and it is repeated till the root of the equation is obtained to a desired degree of accuracy. Normally in such methods we start with a known approximate root and apply an *iterative method* to obtain a root to a desired degree of accuracy.

We present the following methods for obtaining the roots to a desired degree of accuracy of algebraic and transcendental equations.

1. Iteration method (Method of successive approximation)
2. Bisection method (Bolzano method)
3. Newton – Raphson method (Method of tangents)
4. Regula – Falsi method (method false position)
5. Secant method

To locate a root of an equation $f(x) = 0$, the following statement of differential calculus is useful.

If $f(x)$ is continuous in the interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs then the equation $f(x) = 0$ has atleast one root laying between $a$ and $b$.

## 1.3 Iteration Method (Method of successive approximation)

Let $f(x) = 0$ be the given equation, algebraic or transcendental. Suppose the equation can be expressed in the form

$$x = g(x)$$

where $g(x)$ is a continuous function.

Let $x_0$ be an approximate value of the root of the function. Now the value $g(x_0)$ of the function $g(x)$ at $x = x_0$, is the next approximation of the root, called first approximation and we put,

$$x_1 = g(x_0)$$

Similarly, the second, third, fourth, etc, approximations are given by

$$x_2 = g(x_1), \quad x_3 = g(x_2), \quad x_4 = g(x_3), \ldots\ldots, x_n = g(x_{n-1}).$$

The sequence $\{x_n\} = x_0, x_1, x_2, \ldots, x_n, \ldots$ is called the sequence of successive approximations.

The iterative process is said to converge, if the sequence $\{x_n\}$ converges to a root of the equation.

Below we state, without proof, the sufficient condition for the convergence of the sequence of successive approximations.

Let $f(x) = 0$, be the equation, which can be written in the form $x = g(x)$, where $g(x)$ is a continuous function. Let $\alpha$ be a root of $f(x) = 0$ and I be the interval containing $\alpha$. Then for any $x_0 \in I$, the sequence of successive approximations $\{x_n\}$ converges to $\alpha$, if $|g'(x)| < 1$ for all $x \in I$.

**Example 1. Find the real cube root of 15, correct to four significant figures by the method of iteration.**

**Solution :** Let $x = (15)^{1/3} \Rightarrow x^3 = 15 \Rightarrow x^3 - 15 = 0$

Let $f(x) = x^3 - 15$. Clearly $f(2) = -7$ and $f(2.5) = 0.625$.

Thus the real root of the equation $f(x) = 0$ lies between 2 and 2.5.

Now, $x^3 - 15 = 0$ can be written as

$$x = \frac{15 + 20x - x^3}{20} = g(x)$$

Clearly $g(x)$ is a continuous function and further

$$|g'(x)| = \left|1 - \frac{3x^2}{20}\right| < 1$$

in the interval (2, 2.5)

Thus we can take the initial approximation of the root of the equation as $x_0 = 2.5$.

Now, $$x_1 = g(x_0) = \frac{15 + 20(2.5) - (2.5)^3}{20} = 2.47$$

$$x_2 = g(x_1) = \frac{15 + 20(2.47) - (2.47)^3}{20} = 2.466$$

$$x_3 = g(x_2) = \frac{15 + 20(2.466) - (2.466)^3}{20} = 2.4661$$

$$x_4 = g(x_3) = \frac{15 + 20(2.4661) - (2.4661)^3}{20} = 2.4661023.$$

Here $x_3 = x_4$, upto four places of decimals. Thus the cube root of 15 correct to four decimal places is 2.4661.

**Example 2. Find the real root of the equation $x^3 + x + 1 = 0$ correct to three decimal places.**

**Solution :** Let $f(x) = x^3 + x + 1 = 0$.

Clearly $f(x)$ has no positive root. (by applying De'scartis Rule of signs *).

Consider $f(-x) = -x^3 - x + 1 = -(x^3 + x - 1) = -h(x)$.

Thus we shall find a positive root '$a$' of $h(x)$, then $-a$ will be a root of $f(x) = 0$.

Now, $h(0.6) < 0$ and $h(0.7) > 0$

Thus a root of $h(x)$ lies between 0.6 and 0.7. Let the first approximation be $x_0 = 0.65$.

The equation $h(x) = 0$ can be written as

$$x = (1 - x)^{1/3} = g(x)$$

$$\Rightarrow \quad g'(x) = \frac{-1}{3(1 - x)^{2/3}}$$

$$\Rightarrow \quad g'(0.65) = \frac{-1}{3(1 - 0.65)^{2/3}} = -0.687 < 1.$$

Thus the process is convergent

The iterative formula is given by

$$x_{n+1} = g(x_n)$$

Thus, when $n = 0$,

we have $x_1 = g(x_0) = (1 - 0.65)^{1/3} = (0.35)^{1/3} = 0.70$

Similarly, $x_2 = g(x_1) = (1 - 0.70)^{1/3} = (0.30)^{1/3} = 0.68$

$x_3 = g(x_2) = (1 - 0.68)^{1/3} = (0.32)^{1/3} = 0.684$

$x_4 = g(x_3) = (1 - 0.684)^{1/3} = (0.316)^{1/3} = 0.681$

$x_5 = g(x_0) = (1 - 0.681)^{1/3} = (0.3190)^{1/3} = 0.6830$

$x_6 = g(x_5) = (1 - 0.6830)^{1/3} = (0.3170)^{1/3} = 0.6819$

$x_7 = g(x_6) = (1 - 0.6819)^{1/3} = (0.3181)^{1/3} = 0.6826$

$x_8 = g(x_7) = (1 - 0.6826)^{1/3} = (0.3174)^{1/3} = 0.6822.$

Thus the root of $h(x) = 0$, correct to three decimal places is 0.6822.

Thus a real root of $f(x) = 0$ is $-0.6822$.

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Descarte's Rule of signs states that (i) the number of positive roots of the equation $f(x) = 0$, does not exceed the number of changes in the signs of the coefficients of $f(x)$ (ii) the number of negative roots does not exceed the number of changes in the signs of the coefficients of $f(-x)$.

**Example 3. Find the root of the equation $2x = \cos x + 3$ correct to three decimal places, near $x = \frac{\pi}{2}$.**

**Solution :** The given equation can be written as

$$x = \frac{1}{2}(\cos x + 3) = g(x)$$

$$\Rightarrow \qquad g'(x) = \frac{1}{2}(-\sin x)$$

Also $$\left|g'\left(\frac{\pi}{2}\right)\right| = \left|-\frac{1}{2}\right| < 1$$

Now the iterative formula is given by

$$x_{n+1} = g(x_n)$$

Starting with $x_0 = \frac{\pi}{2}$, we obtain successive approximations as

$x_1 = 1.5$, $x_2 = 1.535$, $x_3 = 1.518$, $x_4 = 1.526$,

$x_5 = 1.522$, $x_6 = 1.524$, $x_7 = 1.523$, $x_8 = 1.524$,

Thus the root correct to three decimal places is 1.524.

**Example 4. Find the real root lying between 1 and 2 of the equation $x^3 - 3x + 1 = 0$ using the method of iteration.**

**Solution :** Let $f(x) = x^3 - 3x + 1 = 0$. Now $f(1) = -1$ and $f(2) = 3$

Thus there exists one root between 1 and 2.

$$f(x) = 0 \quad \Rightarrow \quad x^3 = 3x - 1$$

$$\Rightarrow \quad x = (3x - 1)^{1/3} = g(x)$$

Also, $$g'(x) = \frac{1}{(3x-1)^{2/3}}$$

Clearly, $|g'(x)| < 1$ for all $x \in (1, 2)$

Thus the sequence of successive approximations is convergent, with $x_0 = 2$.

Now, $$x_1 = g(x_0) = [3(2) - 1]^{1/3} = (5)^{1/3} = 1.7100$$

$$x_2 = g(x_1) = [3(1.7100) - 1]^{1/3} = (4.13)^{1/3} = 1.6044$$

$$x_3 = g(x_2) = [3(1.6044) - 1]^{1/3} = (3.8132)^{1/3} = 1.5623$$

$$x_4 = g(x_3) = [3(1.5623) - 1]^{1/3} = (3.6869)^{1/3} = 1.5449$$

$$x_5 = g(x_4) = [3(1.5449) - 1]^{1/3} = (3.6347)^{1/3} = 1.5375$$

$$x_6 = g(x_5) = [3(1.5375) - 1]^{1/3} = (3.6125)^{1/3} = 1.5344$$

$$x_7 = g(x_6) = [3(1.5344) - 1]^{1/3} = (3.6032)^{1/3} = 1.5331$$

$$x_8 = g(x_7) = [3(1.5331) - 1]^{1/3} = (3.5993)^{1/3} = 1.5325$$

$$x_9 = g(x_8) = [3(1.5325) - 1]^{1/3} = (3.5975)^{1/3} = 1.5323$$

Here $x_8 = x_9$, upto 3 places of decimals.

Thus an approximate root of the equation is 1.532.

**Example 5. Find a real root of the equation $\cos x = 3x - 1$ correct to 4 decimal places by iteration method.**

**Solution :** Let $f(x) = \cos x - 3x + 1 = 0$.

Now, $f(0) = 2 > 0$ and $f\left(\frac{\pi}{2}\right) = 1 - 3\left(\frac{\pi}{2}\right) = 1 - 4.7123 < 0$

Thus there exists a root between 0 and $\frac{\pi}{2}$.

Now, $f(x) = 0 \Rightarrow \cos x - 3x + 1 = 0$

$$\Rightarrow x = \frac{1}{3}(1 + \cos x) = g(x)$$

Also, $g'(x) = -\frac{1}{3}\sin x$

Clearly, $|g'(x)| < 1$ for all $x \in \left(0, \frac{\pi}{2}\right)$

Thus the sequence of successive approximations is convergent, with $x_0 = 0$.

Now,

$$x_1 = g(x_0) = \frac{1}{3}(1 + \cos 0) = 0.6667$$

$$x_2 = g(x_1) = \frac{1}{3}[1 + \cos(0.6667)] = 0.5953$$

$$x_3 = g(x_2) = \frac{1}{3}[1 + \cos(0.5953)] = 0.6093$$

$$x_4 = g(x_3) = \frac{1}{3}[1 + \cos(0.6093)] = 0.6067$$

$$x_5 = g(x_4) = \frac{1}{3}[1 + \cos(0.6067)] = 0.6072$$

$$x_6 = g(x_5) = \frac{1}{3}[1 + \cos(0.6072)] = 0.6071$$

$$x_7 = g(x_6) = \frac{1}{3}[1 + \cos(0.6071)] = 0.6071$$

Thus the root is 0.6071 correct to 4 decimal places.

**Example 6. Find a real root lying between 2 and 3 of the equation $3x - \log_{10}x = 6$ using the method of iteration.**

**Solution :** Let $f(x) = 3x - \log_{10}x - 6$.

Now, $f(2) = 6 - \log_{10}2 - 6 = -0.3010 < 0$

$f(3) = 9 - \log_{10}3 - 6 = 2.5229 > 0$

Thus a root of the equation $f(x) = 0$ lies between 2 and 3.

Now, $f(x) = 0 \Rightarrow 3x - \log_{10}x - 6 = 0.$

$$\Rightarrow x = \frac{1}{3}(6 + \log_{10}x) = g(x)$$

Also $g'(x) = \frac{1}{3}\left(\frac{\log_{10}e}{x}\right)$ $\left[\text{since } \log_{10}x = \frac{\log_e x}{\log_e 10} = \log_{10}e \cdot \log_e x\right]$

Now, $g'(x) = \frac{0.4343}{3}\left|\frac{1}{x}\right|$ $\left[\text{since } \log_{10}e = 0.4343\right]$

$< 1, \text{ for } x \in (2, 3)$

We shall take $x_0 = 2$

$$x_1 = g(x_0) = \frac{1}{3}(6 + \log_{10}2) = \frac{6.3010}{3} = 2.1003$$

$$x_2 = g(x_1) = \frac{1}{3}(6 + \log_{10}2.1003) = \frac{6.3223}{3} = 2.1074$$

$$x_3 = g(x_2) = \frac{1}{3}(6 + \log_{10}2.1074) = \frac{6.3238}{3} = 2.1079$$

$$x_4 = g(x_3) = \frac{1}{3}(6 + \log_{10}2.1079) = \frac{6.3239}{3} = 2.1080$$

$$x_5 = g(x_4) = \frac{1}{3}(6 + \log_{10}2.1080) = \frac{6.3239}{3} = 2.1080$$

Thus the approximate value of the root 2.108.



## EXERCISES

Solve the following equations using iterative method

1. $x^3 + x^2 - 1 = 0$
2. $x^3 + x^2 - 100 = 0$
3. $\cos x = 3x - 1$
4. $3x - \sqrt{1 + + \sin x} = 0$
5. $1 + \sin x - 2x = 0$
6. $2x - \log_{10}x = 7$
7. $3x - \cos x - 2 = 0$
8. $e^x - 3x = 0$
9. $3x + \sin x = e^x$
10. $x^3 + 2x^2 + 10x = 20$
11. Find the negative root of $x^3 - 2x + 5 = 0$

## ANSWERS

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 0.75488 | 2. | 4.3311 | 3. | 0.607 | 4 | 0.39185 |
| 5. | 1.497 | 6. | 3.7893 | 7. | 0.879 | 8. | 0.6190 |

9. 0.3604 10. 1.3688 11. $-2.0945$

## 1.4 Bisection Method (Bolzano method)

This method describes a procedure to locate the root of the equation $f(x) = 0$, between two real numbers $a$ and $b$, $a < b$. Let $f(x)$ be a continuous function, (algebraic or transcendental) in the interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs. Then we know that there exists a root between $a$ and $b$. Let $f(a) < 0$ and $f(b) > 0$. We now bisect the interval $(a, b)$ and denote the mid point by $x_1$, so that

$$x_1 = \frac{a+b}{2}$$

If $f(x_1) = 0$, then $x_1$ is a root of $f(x) = 0$

If $f(x_1) \neq 0$, then the root lies between $a$ and $x_1$ or $x_1$ and $b$, depending on $f(x_1) > 0$ or $f(x_1) < 0$. Let $f(x_1) > 0$, then the root lies between $a$ and $x_1$ (If $f(x_1) < 0$ then the root lies between $x_1$ and $b$). Now bisect the interval $(a, x_1)$ at $x_2 = \frac{a+x_1}{2}$. Again if $f(x_2) = 0$, then $x_2$ is a root. Otherwise the root lies between $a$ and $x_2$ or $x_2$ and $x_1$ depending on $f(x_2) > 0$ or $f(x_2) < 0$. Now choose that interval in which the root lies and bisect and continue the process until the root is found to the desired accuracy.

By repeating this *bisection* procedure, we obtain a sequence of nested intervals, say $I_0 \supset I_1 \supset I_2 \supset$........ such that each subinterval contains the root. After repeating the bisection procedure $r$ times, we either find the root or find the interval $I_r$ of length $\frac{b-a}{2^r}$ which contains the root. The mid point of last subinterval is taken as the desired approximation to the root. Further this root has error not greater than one half of the length of the interval of which, it is the mid point.

Even though this method is simple, the convergence of sequence of approximations is slow but sure.

**Example 1. Find a real root of the equation $x^3 - x - 1 = 0$ lying between 1 and 2 correct to three places of decimal by using bisection method.**

**Solution :** Let $f(x) = x^3 - x - 1$. Clearly $f(1) = -1$ and $f(2) = 5$

Thus $f(x) = 0$ has one root lying between 1 and 2.

The first approximation is

$$x_1 = \frac{a+b}{2} = 1.5$$

Now, $f(x_1) = f(1.5) = \frac{7}{8} > 0$. Also $f(1) < 0$.

Thus the root lies between 1 and 1.5.

Let $a = 1$ and $b = 1.5$

The second approximation is

$$x_2 = \frac{a+b}{2} = \frac{1+1.5}{2} = 1.25$$

Now, $f(x_2) = f(1.25) = \frac{19}{64} < 0$. Also $f(1.5) > 0$.

Thus the root lies between 1.25 and 1.5.

Let $a = 1.25$ and $b = 1.5$

The third approximation is

$$x_3 = \frac{a+b}{2} = \frac{1.25 + 1.5}{2} = 1.375$$

Now, $f(x_3) = \frac{115}{512} > 0$. Also $f(1.25) < 0$.

Thus the root lies between 1.25 and 1.375.

This process is repeated and the calculation of successive approximations are shown in the following table.

| $i$ | $a$ | $b$ | $x_i = \frac{a+b}{2}$ | $f(x_i)$ |
|---|---|---|---|---|
| 1 | 1 | 2 | 1.5 | 0.8750 |
| 2 | 1 | 1.5 | 1.25 | − 0.29688 |
| 3 | 1.25 | 1.5 | 1.375 | 0.22461 |
| 4 | 1.25 | 1.375 | 1.3125 | − 0.051514 |
| 5 | 1.375 | 1.3125 | 1.3438 | 0.08283 |
| 6 | 1.3125 | 1.3438 | 1.3282 | 0.01489 |
| 7 | 1.3125 | 1.3282 | 1.3204 | − 0.01834 |
| 8 | 1.3204 | 1.3282 | 1.3243 | − 0.0018 |
| 9 | 1.3243 | 1.3282 | 1.3263 | 0.0068 |
| 10 | 1.3243 | 1.3263 | 1.3253 | 0.0025 |
| 11 | 1.3243 | 1.3253 | 1.3248 | 0.0004 |
| 12 | 1.3243 | 1.3248 | 1.32455 | |

We observe $x_{11}$ and $x_{12} = 1.324$ correct to 3 places of decimals. Hence the required root is 1.324.

**Example 2. Find a real positive root of the equation $x^3 - 7x + 5 = 0$ by using bisection method, correct to three places of decimal.**

**Solution :** Let $f(x) = x^3 - 7x + 5$.

We observe $f(0) = 5 > 0$ and $f(1) = -1 < 0$

Thus $f(x) = 0$ has one root lying between $a = 0$ and $b = 1$.

Now the first approximation is

$$x_1 = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$$

Now, $f(x_1) = f(0.5) = (0.5)^3 - 7(0.5) + 5 = 1.625 > 0$. Also $f(1) < 0$.

Thus the root lies between $a = 1$ and $b = 0.5$.

Further calculations are shown in the following table.

| $i$ | $a$ | $b$ | $x_i = \frac{a+b}{2}$ | $f(x_i)$ |
|---|---|---|---|---|
| 1 | 0 | 1 | 0.5 | |
| 2 | 1 | 0.5 | 0.75 | 1.625 |
| 3 | 1 | 0.75 | 0.875 | 0.17187 |
| 4 | 0.875 | 0.75 | 0.8125 | – 0.45507 |
| 5 | 0.8125 | 0.75 | 0.7812 | – 0.15112 |
| 6 | 0.8125 | 0.7812 | 0.7968 | 0.00799 |
| 7 | 0.7968 | 0.7812 | 0.7890 | – 0.07171 |
| 8 | 0.7890 | 0.7812 | 0.7851 | – 0.03183 |
| 9 | 0.7851 | 0.7812 | 0.7831 | – 0.01177 |
| 10 | 0.7831 | 0.7812 | 0.7822 | – 0.0157 |
| 11 | 0.7831 | 0.7822 | 0.7826 | 0.0031 |

We observe $x_{10}$ and $x_{11}$ are equal to 0.782 correct to 3 places of decimals. Hence the required root is 0.782.

**Example 3. Find the positive real root of the equation $x \log_{10} x = 1.2$, using bisection method in four iterations.**

**Solution :** Let $f(x) = x\log_{10}x - 1.2.$

Now, $f(2) = 2\log_{10}2 - 1.2 = 0.6020 - 1.2 = -0.598 < 0$

and $f(3) = 3\log_{10}3 - 1.2 = 1.4314 - 1.2 = 0.2314 > 0$

Thus the root lies between 2 and 3.

The first approximation is

$$x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$$

Now, $f(2.5) = (2.5)\log_{10}2.5 - 1.2 = 0.99485 - 1.2 = -2051 < 0.$

Thus the root lies between 2.5 and 3.

Let $a = 2.5$ and $b = 3$

The second approximation is

$$x_2 = \frac{a+b}{2} = \frac{2.5+3}{2} = 2.75$$

Now, $f(2.75) = (2.75)\log_{10}2.75 - 1.2 = (2.75)(0.4393) - 1.2 = 0.008 > 0.$

Thus the root lies between 2.5 and 2.75.

Let $a = 2.5$ and $b = 2.75$